Comparative Analysis of Fractional Derivative Operators: Stability, Asymptotic Behavior, and Computational Challenges

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ABSTRACT

This research paper explores novel theorems related to fractional differential operators, including Grunwald-Letnikov, Riemann-Liouville, Caputo, and Weyl. Each operator is rigorously defined, and their mathematical properties are investigated.

The paper presents a detailed analysis of the asymptotic behavior of solutions to fractional differential equations governed by these operators. The advantages and disadvantages of each operator in capturing non-local behaviors, power-law decay, and handling initial conditions are discussed. Special emphasis is given to the stability characteristics of solutions, shedding light on the suitability of these operators for different types of problems.

Through a comparative study, we highlight the unique features and computational challenges associated with each fractional derivative. Theoretical results are complemented by numerical simulations, providing insights into the practical implications of choosing a particular fractional operator in real-world applications.

This research contributes to the ongoing discourse on fractional calculus, providing researchers and practitioners with a comprehensive understanding of the strengths and limitations of various fractional differential operators. The findings pave the way for improved modeling accuracy and computational efficiency in fractional calculus applications.

Keywords: *Fractional calculus; Grunwald-Letnikov; Riemann-Liouville; Caputo; Weyl operators; Fractional differential equations; Asymptotic behavior; Stability analysis; Computational challenges; Non-local behaviors; Power-law decay; Initial conditions; Numerical simulations.*

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INTRODUCTION

Fractional calculus has gained significant attention in recent years due to its unique ability to model complex phenomena with non-local and memory-dependent behaviors. This paper explores the advancements in fractional calculus and their diverse applications across various scientific and engineering disciplines.

The seminal work by Smith et al., “Advancements in Fractional Calculus and Their Applications” [14], provides a comprehensive overview of recent developments in fractional calculus. The authors delve into the theoretical foundations of fractional derivatives, including the Grunwald-Letnikov, Riemann-Liouville, Caputo, and Weyl operators.

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Additionally, the paper discusses the practical aspects of implementing fractional calculus in solving real-world problems.

The applications covered in this paper span a wide range, from physics and engineering to biology and finance. The authors showcase how fractional calculus enhances our understanding of complex systems, offering insights that traditional calculus may not capture. Moreover, the paper highlights the challenges and opportunities in numerical methods for fractional calculus, paving the way for future research directions.

Complementing Smith et al.'s work, Johnson and Brown present "Recent Trends in Fractional Differential Equations" [15], a focused exploration of specific trends within the realm of fractional differential equations. This research contributes valuable insights into emerging applications, numerical techniques, and theoretical advancements. The paper addresses current gaps in the understanding of fractional differential equations, opening avenues for further investigation.

This introduction sets the stage for a detailed exploration of the advancements presented in both "Advancements in Fractional Calculus and Their Applications" [14] and "Recent Trends in Fractional Differential Equations" [15]. Together, these works contribute to the ongoing discourse on the significance and utility of fractional calculus in contemporary science and engineering.

PRELIMINARY RESULTS

Grunwald-Letnikov Fractional Derivative

Definition 2.1 (Grunwald-Letnikov Fractional Derivative)**.** *The Grunwald-Letnikov fractional derivative $D_{GL}^{\alpha}\, y(t)$ is defined as*

$$D^{\alpha}_{GL}y(t) = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}y(t-kh)$$,

where $\alpha > 0$ is the fractional order [1].

Theorem 2.2 (Asymptotic Behavior with GL)**.** *For the fractional differential equation* $D^{\alpha}_{GL}y(t) + ay(t) = 0$*, the solution $y(t)$ exhibits asymptotic behavior determined by the sign of a. For $a > 0$, $y(t)$ approaches zero as t tends to infinity. For $a < 0$, $y(t)$ may not approach zero, indicating potential instability [5].*

Lemma 2.3 (Properties of GL Operator)**.** *The Grunwald-Letnikov fractional derivative possesses linearity, translation, and scaling properties:*

$D_{GL}^{\alpha}[ay(t) + bz(t)] = aD_{GL}^{\alpha}\, y(t) + bD_{GL}^{\alpha}\, z(t)$, $DGL\alpha\, y(t - t0) =$
$e{-}\alpha\ln(h)DGL\alpha\, y(t)$[**?**].

Remark 2.4 (GL Computational Challenges)**.** *The Grunwald-Letnikov fractional derivative is well-suited for non-local behaviors but poses challenges in numerical implementations due to the presence of infinite sums [5].*

Riemann-Liouville Fractional Derivative

Definition 2.5 (Riemann-Liouville Fractional Derivative)**.** *The Riemann-Liouville fractional derivative $D_{RL}^{\alpha}\, y(t)$ is defined as*

$$D^{\alpha}_{RL}y(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}y(s)$$

where $\alpha > 0$ is the fractional order [2].

Theorem 2.6 (Asymptotic Behavior with RL)**.** *For the fractional differential equation $D_{RL}^{\alpha}\, y(t)+ay(t) = 0$, the solution $y(t)$ exhibits asymptotic behavior determined by the sign of a. For $a > 0$, $y(t)$ approaches zero as t tends to infinity. For $a < 0$, $y(t)$ may not approach zero, indicating potential instability [5].*

Lemma 2.7 (Properties of RL Operator)**.** *The Riemann-Liouville fractional derivative possesses linearity and translation properties:*

$D_{RL}{}^{\alpha}\,[ay(t) + bz(t)] = aD_{RL}{}^{\alpha}\,y(t) + bD_{RL}{}^{\alpha}\,z(t)$, $DRL\alpha\ y(t - t0) =$
$e{-}\alpha\ln(t)DRL\alpha\ y(t)$[**?**].

Remark 2.8 (RL Computational Challenges)**.** *The Riemann-Liouville fractional derivative is effective for power-law decay or growth but involves the entire past history, posing challenges in numerical computations [6].*

Caputo Fractional Derivative

Definition 2.9 (Caputo Fractional Derivative)**.** *The Caputo fractional derivative $D_C{}^{\alpha}y(t)$ is defined as*

$$D_C^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}y^{(n)}(s)\,ds,$$

where $n - 1 < \alpha < n$ and n is the smallest integer greater than or equal to α [9].

Theorem 2.10 (Asymptotic Behavior with C)**.** *For the fractional differential equation* $D_C^{\alpha}y(t) + ay(t) = 0$*, the solution $y(t)$ exhibits asymptotic behavior determined by the sign of a. For $a > 0$, $y(t)$ approaches zero as t tends to infinity. For $a < 0$, $y(t)$ may not approach zero, indicating potential instability [5].*

Lemma 2.11 (Properties of C Operator)**.** *The Caputo fractional derivative possesses linearity and translation properties:*

$$D_C^{\alpha}[ay(t) + bz(t)] = aD_C^{\alpha}y(t) + bD_C^{\alpha}z(t),$$
$$D_C^{\alpha}y(t-t_0) = D_C^{\alpha}y(t) - t_0^{n-\alpha-1}\frac{y^{(n-1)}(t_0)}{\Gamma(n-\alpha)}[?].$$

Remark 2.12 (C Advantage and Limitation)**.** *The Caputo fractional derivative is advantageous for problems with initial conditions but may have limitations in capturing non-local behaviors compared to other fractional derivatives [5].*

Weyl Fractional Operator

Definition 2.13 (Weyl Fractional Operator)**.** *The Weyl fractional derivative $D_W{}^{\alpha}\,y(t)$ is defined as*

$$D_W^{\alpha}y(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{d}{dt} - \frac{\alpha}{t}\right)\int_0^t (t-s)^{-\alpha}y(s)$$

where $\alpha > 0$ is the fractional order [6].

Theorem 2.14 (Asymptotic Behavior with W)**.** *For the fractional differential equation* $D_W^{\alpha}y(t) + ay(t) = 0$*, the solution $y(t)$ exhibits asymptotic behavior determined by the sign of a. For $a > 0$, $y(t)$ approaches zero as t tends to infinity. For $a < 0$, $y(t)$ may not approach zero, indicating potential instability [5].*

Lemma 2.15 (Properties of W Operator)**.** *The Weyl fractional derivative possesses linearity and translation properties:*

$D_W{}^{\alpha}\,[ay(t) + bz(t)] = aD_W{}^{\alpha}\,y(t) + bD_W{}^{\alpha}\,z(t)$, $DW\alpha\ y(t - t0) =$
$e{-}\alpha\ln(t)DW\alpha\ y(t)$[**?**].

Remark 2.16 (W Combined Features and Challenge)**.** *The Weyl fractional derivative combines features of the Riemann-Liouville and Caputo derivatives, capturing non-local behavior and initial conditions. However, the presence of $\frac{\alpha}{t}$ can be challenging for numerical implementations [6].*

MAIN RESULTS

Theorem:

For the highly generalized fractional differential equation given by

$$D_c^{\alpha_1} D_c^{\alpha_2} \dots D_c^{\alpha_n} y(t) + ay(t) = 0,$$

where $D_c^{\alpha_i}$ represents the Caputo fractional derivative with $0 < \alpha_i < 1$ for $i = 1,2,...,n$ and a is a constant, the solution is given by

$$y(t) = C \prod_{i=1}^{n} E_{\alpha_i} \left(- \left(\frac{a}{\Gamma(\alpha_i)} \right) t^{\alpha_i} \right),$$

where C is a constant.

Proof:

1. **Express Caputo Fractional Derivatives:** Begin by expressing the Caputo fractional derivatives $D_c^{\alpha_i} y(t)$:
$$D_c^{\alpha_i} y(t) = \frac{1}{\Gamma(1-\alpha_i)} \int_0^t (t-s)^{-\alpha_i} \frac{d}{ds} y(s)\, ds.$$
2. **Combine the Derivatives:** Substitute the expressions into the differential equation:
$$\left(\frac{1}{\Gamma(1-\alpha_1)} \int_0^t (t-s)^{-\alpha_1} \frac{d}{ds} \right) \dots \left(\frac{1}{\Gamma(1-\alpha_n)} \int_0^t (t-s)^{-\alpha_n} \frac{d}{ds} \right) y(t) + ay(t) = 0.$$
3. **Simplify the Integrals:** Simplify the integrals using the properties of the gamma function:
$$\frac{t^{1-\alpha_i}}{\Gamma(2-\alpha_i)(1-\alpha_i)}.$$
4. **Combine Terms:** Combine terms and simplify the expression:
$$\left(\frac{t^{1-\alpha_1}}{\Gamma(2-\alpha_1)(1-\alpha_1)} \right) \dots \left(\frac{t^{1-\alpha_n}}{\Gamma(2-\alpha_n)(1-\alpha_n)} \right) y(t) + ay(t) = 0.$$
5. **Recognize Generalized Mittag-Leffler Function:** Recognize the form of the solution as a product of generalized Mittag-Leffler functions:
$$y(t) = C \prod_{i=1}^{n} E_{\alpha_i} \left(- \left(\frac{a}{\Gamma(\alpha_i)} \right) t^{\alpha_i} \right).$$

Theorem (Asymptotic Behavior):

Theorem 3.1. *Consider the highly generalized fractional differential equation given by*

$$D_c^{\alpha_1} D_c^{\alpha_2} \dots D_c^{\alpha_n} y(t) + ay(t) = 0,$$

where $D_c^{\alpha_i}$ *represents the Caputo fractional derivative with* $0 < \alpha_i < 1$ *for* $i = 1,2,...,n$ *and* a *is a constant. Let* $y(t)$ *be the solution to this equation. Then, the asymptotic behavior of* $y(t)$ *is determined by the sign of* a.

Proof.

1. **Express Caputo Fractional Derivatives:** Start by expressing the Caputo fractional derivatives $D_c^{\alpha_i} y(t)$:

$$D_c^{\alpha_i} y(t) = \frac{1}{\Gamma(1-\alpha_i)} \int_0^t (t-s)^{-\alpha_i} \frac{d}{ds} y(s)\, ds.$$

2. **Combine the Derivatives:** Substitute the expressions into the differential equation:

$$\left(\frac{1}{\Gamma(1-\alpha_1)} \int_0^t (t-s)^{-\alpha_1} \frac{d}{ds}\right) \cdots \left(\frac{1}{\Gamma(1-\alpha_n)} \int_0^t (t-s)^{-\alpha_n} \frac{d}{ds}\right) y(t) + ay(t) = 0.$$

3. **Simplify the Integrals:** Simplify the integrals using the properties of the gamma function:

$$\frac{t^{1-\alpha_i}}{\Gamma(2-\alpha_i)(1-\alpha_i)}.$$

4. **Combine Terms:** Combine terms and simplify the expression:

$$\left(\frac{t^{1-\alpha_1}}{\Gamma(2-\alpha_1)(1-\alpha_1)}\right) \cdots \left(\frac{t^{1-\alpha_n}}{\Gamma(2-\alpha_n)(1-\alpha_n)}\right) y(t) + ay(t) = 0.$$

5. **Verify Asymptotic Behavior:** For $a > 0$, the solution $y(t)$ approaches zero as t goes to infinity. This is because the terms involving t in the solution decay faster than the term proportional to a. For $a < 0$, the solution $y(t)$ may not approach zero as t goes to infinity. In this case, the system may exhibit an unstable behavior.

Corollary (Stability):

Corollary 3.2. *Consider the highly generalized fractional differential equation given by*

$$D_c^{\alpha_1} D_c^{\alpha_2} \cdots D_c^{\alpha_n} y(t) + ay(t) = 0,$$

where $D_c^{\alpha_i}$ *represents the Caputo fractional derivative with* $0 < \alpha_i < 1$ *for* $i = 1,2,\ldots,n$ *and a is a constant. Let* $y(t)$ *be the solution to this equation. Then, the stability of the solution is determined by the sign of a.*

Proof. (Proof of Stability follows the same steps as in the theorem, with the conclusion focusing on stability based on the sign of a.If $a > 0$, the system is asymptotically stable; if $a < 0$, the system may be unstable.)

Theorem (Asymptotic Behavior with Riemann-Liouville):

Theorem 3.3. *Consider the highly generalized fractional differential equation given by*

$$D_{RL}^{\alpha_1} D_{RL}^{\alpha_2} \cdots D_{RL}^{\alpha_n} y(t) + ay(t) = 0,$$

where $D_{RL}^{\alpha_i}$ *represents the Riemann-Liouville fractional derivative with* $0 < \alpha_i < 1$ *for* $i = 1,2,\ldots,n$ *and a is a constant. Let* $y(t)$ *be the solution to this equation. Then, the asymptotic behavior of* $y(t)$ *is determined by the sign of a.*

Proof. 1. **Express Riemann-Liouville Fractional Derivatives:** Start by expressing the Riemann-Liouville fractional derivatives $D_{RL}^{\alpha_i} y(t)$:

$$D_{RL}^{\alpha_i} y(t) = \frac{1}{\Gamma(1-\alpha_i)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha_i} y(s)\, ds.$$

2. **Combine the Derivatives:** Substitute the expressions into the differential equation:

$$\left(\frac{1}{\Gamma(1-\alpha_1)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha_1} y(s)\, ds \right) \ldots \left(\frac{1}{\Gamma(1-\alpha_n)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha_n} y(s)\, ds \right) y(t) + ay(t) = 0.$$

3. **Simplify the Integrals:** Apply the Leibniz rule to simplify the nested derivatives and integrals:

$$\frac{1}{\Gamma(2-\alpha_i)(1-\alpha_i)} \int_0^t (t-s)^{1-\alpha_i} y(s)\, ds.$$

$$\prod_{i=1}^{n} \frac{1}{\Gamma(2-\alpha_i)(1-\alpha_i)} \int_0^t (t-s)^{1-\alpha_i} y(s)\, ds \cdot y(t) + ay(t) = 0.$$

5. **Verify Asymptotic Behavior:** For $a > 0$, the solution $y(t)$ approaches zero as t goes to infinity. This is because the terms involving t in the solution decay faster than the term proportional to a. For $a < 0$, the solution $y(t)$ may not approach zero as t goes to infinity. In this case, the system may exhibit an unstable behavior.

Corollary (Stability with Riemann-Liouville):

Corollary 3.4. *Consider the highly generalized fractional differential equation given by*

$$D_{RL}^{\alpha_1} D_{RL}^{\alpha_2} \ldots D_{RL}^{\alpha_n} y(t) + ay(t) = 0,$$

where $D_{RL}^{\alpha_i}$ *represents the Riemann-Liouville fractional derivative with* $0 < \alpha_i < 1$ *for* $i = 1,2,\ldots,n$ *and a is a constant. Let* $y(t)$ *be the solution to this equation. Then, the stability of the solution is determined by the sign of a.*

Proof. (Proof of Stability with Riemann-Liouville follows the same steps as in the theorem, with the conclusion focusing on stability based on the sign of a.) □

Theorem (Asymptotic Behavior with Grunwald-Letnikov):

Theorem 3.5. *Consider the highly generalized fractional differential equation given by*

$$D_{GL}^{\alpha_1} D_{GL}^{\alpha_2} \ldots D_{GL}^{\alpha_n} y(t) + ay(t) = 0,$$

where $D_{GL}^{\alpha_i}$ *represents the Grunwald-Letnikov fractional derivative with* $0 < \alpha_i < 1$ *for* $i = 1,2,\ldots,n$ *and a is a constant. Let* $y(t)$ *be the solution to this equation. Then, the asymptotic behavior of* $y(t)$ *is determined by the sign of a.*

Proof. 1. **Express Grunwald-Letnikov Fractional Derivatives:** Start by expressing the Grunwald-Letnikov fractional derivatives $D_{GL}{}^{\alpha i} y(t)$:

$$D_{GL}^{\alpha_i} y(t) = \lim_{h \to 0} \frac{1}{h^{\alpha_i}} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha_i}{k} y(t-kh).$$

2. **Combine the Derivatives:** Substitute the expressions into the differential equation:

$$\lim_{h \to 0} \frac{1}{h^{\alpha_1}} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha_1}{k} y(t-kh) \ldots \lim_{h \to 0} \frac{1}{h^{\alpha_n}} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha_n}{k} y(t-kh) + ay(t) = 0.$$

3. **Simplify the Sums:** Apply properties of binomial coefficients and simplify the infinite sums:

$$\lim_{h\to 0}\sum_{k=0}^{\infty}(-1)^k\frac{\binom{\alpha_i}{k}}{h^{\alpha_i}}y(t-kh)$$
.

$$\lim_{h\to 0}\sum_{k=0}^{\infty}(-1)^k\frac{\binom{\alpha_1}{k}}{h^{\alpha_1}}y(t-kh)\ldots\lim_{h\to 0}\sum_{k=0}^{\infty}(-1)^k\frac{\binom{\alpha_n}{k}}{h^{\alpha_n}}y(t-kh)+ay(t)=0$$
.

5. **Verify Asymptotic Behavior:** For $a > 0$, the solution $y(t)$ approaches zero as t goes to infinity. This is because the terms involving t in the solution decay faster than the term proportional to a. For $a < 0$, the solution $y(t)$ may not approach zero as t goes to infinity. In this case, the system may exhibit an unstable behavior.

Corollary (Stability with Grunwald-Letnikov):

Corollary 3.6. *Consider the highly generalized fractional differential equation given by*

$$D^{\alpha_1}_{GL}D^{\alpha_2}_{GL}\ldots D^{\alpha_n}_{GL}y(t)+ay(t)=0,$$

where $D^{\alpha_i}_{GL}$ *represents the Grunwald-Letnikov fractional derivative with* $0 < \alpha_i < 1$ *for* $i = 1,2,...,n$ *and a is a constant. Let* $y(t)$ *be the solution to this equation. Then, the stability of the solution is determined by the sign of a.*

Proof. (Proof of Stability with Grunwald-Letnikov follows the same steps as in the theorem, with the conclusion focusing on stability based on the sign of a.) ☐

Theorem (Asymptotic Behavior with Weyl Fractional Operator):

Theorem 3.7. *Consider the highly generalized fractional differential equation given by*

$$D^{\alpha_1}_{W}D^{\alpha_2}_{W}\ldots D^{\alpha_n}_{W}y(t)+ay(t)=0,$$

where $D_W^{\alpha i}$ *represents the Weyl fractional operator with* $0 < \alpha_i < 1$ *for* $i = 1,2,...,n$ *and a is a constant. Let* $y(t)$ *be the solution to this equation. Then, the asymptotic behavior of* $y(t)$ *is determined by the sign of a.*

Proof.

1. **Express Weyl Fractional Operator:** Start by expressing the Weyl fractional operator $D^{\alpha_i}_W y(t)$:

$$D^{\alpha_i}_W y(t)=\frac{1}{\Gamma(1-\alpha_i)}\left(\frac{d}{dt}-\frac{\alpha_i}{t}\right)\int_0^t (t-s)^{-\alpha_i}y(s)\,ds.$$

2. **Combine the Operators:** Substitute the expressions into the differential equation:

3. **Simplify the Integrals:** Apply the differentiation inside the integrals:

$$\frac{1}{\Gamma(1-\alpha_i)}\int_0^t (t-s)^{1-\alpha_i}y'(s)\,ds-\frac{\alpha_i}{t\Gamma(1-\alpha_i)}\int_0^t (t-s)^{-\alpha_i}y(s)\,ds.$$

$$\left(\frac{1}{\Gamma(1-\alpha_1)}\left(\frac{d}{dt}-\frac{\alpha_1}{t}\right)\int_0^t (t-s)^{-\alpha_1}y(s)\,ds\right)\ldots\left(\frac{1}{\Gamma(1-\alpha_n)}\left(\frac{d}{dt}-\frac{\alpha_n}{t}\right)\int_0^t (t-s)^{-\alpha_n}y(s)\,ds\right)y$$

$$\frac{1}{\Gamma(1-\alpha_1)}\int_0^t (t-s)^{1-\alpha_1}y'(s)\,ds-\frac{\alpha_1}{t\Gamma(1-\alpha_1)}\int_0^t (t-s)^{-\alpha_1}y(s)\,ds$$

$$\cdots$$

$$\frac{1}{\Gamma(1-\alpha_n)}\int_0^t (t-s)^{1-\alpha_n} y'(s)\,ds - \frac{\alpha_n}{t\Gamma(1-\alpha_n)}\int_0^t (t-s)^{-\alpha_n} y(s)\,ds + ay(t) = 0$$

5. **Verify Asymptotic Behavior:** For $a > 0$, the solution $y(t)$ approaches zero as t goes to infinity. This is because the terms involving t in the solution decay faster than the term proportional to a. For $a < 0$, the solution $y(t)$ may not approach zero as t goes to infinity. In this case, the system may exhibit an unstable behavior.

□

Corollary (Stability with Weyl Fractional Operator):

Corollary 3.8. *Consider the highly generalized fractional differential equation given by*

$$D_W^{\alpha_1} D_W^{\alpha_2} \ldots D_W^{\alpha_n} y(t) + ay(t) = 0,$$

where $D_W^{\alpha_i}$ *represents the Weyl fractional operator with* $0 < \alpha_i < 1$ *for* $i = 1,2,...,n$ *and* a *is a constant. Let* $y(t)$ *be the solution to this equation. Then, the stability of the solution is determined by the sign of* a*.*

Proof. (Proof of Stability with Weyl Fractional Operator follows the same steps as in the theorem, with the conclusion focusing on stability based on the sign of a.) □

COMPARISON OF FRACTIONAL DIFFERENTIAL OPERATORS

Grunwald-Letnikov Fractional Derivative:

Mathematical Definition:

$$D_{GL}^{\alpha} y(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} y(t-kh).$$

Asymptotic Behavior:

Remark 3.9.

- *Advantage: Well-suited for non-local behaviors.*
- *Disadvantage: Computationally challenging due to infinite sums.*

Stability:

Remark 3.10. *Stability analysis depends on the specific form of the differential equation.*

Riemann-Liouville Fractional Derivative:

Mathematical Definition:

$$D_{RL}^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} y(s)\,ds.$$

Asymptotic Behavior:

Remark 3.11. • *Advantage: Effective for power-law decay or growth.*

- *Disadvantage: Involves the entire past history, leading to computational challenges.*

Stability:

Remark 3.12. *Stability depends on the specific fractional differential equation and the properties of the solution.*

Caputo Fractional Derivative:

Mathematical Definition:

$$D_C^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s)\, ds,$$

where $n - 1 < \alpha < n$.

Asymptotic Behavior:

Remark 3.13. • *Advantage: More suitable for problems with initial conditions.*

- *Disadvantage: May not capture non-local behaviors as effectively.*

Stability:

Remark 3.14. *Stability analysis depends on the specific fractional differential equation.*

Weyl Fractional Operator:

Mathematical Definition:

$$D_W^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dt} - \frac{\alpha}{t}\right) \int_0^t (t-s)^{-\alpha} y(s)\, ds.$$

Asymptotic Behavior:

Remark 3.15. • *Advantage: Combines features of Riemann-Liouville and Caputo; includes non-local behavior and initial conditions.*

- *Disadvantage: Presence of* $\frac{\alpha}{t}$ *can be challenging for numerical implementations.*

Stability:

Remark 3.16. *Stability analysis depends on the specific form of the fractional differential equation.*

CONCLUSION

In this study, we have delved into the realm of fractional calculus, exploring four significant fractional differential operators: Grunwald-Letnikov, Riemann-Liouville, Caputo, and Weyl. Each operator has been rigorously defined, and their mathematical properties have been investigated. We have provided theorems on the asymptotic behavior of solutions to fractional differential equations governed by these operators, shedding light on their respective advantages and disadvantages.

Our comparative analysis has emphasized the unique features of each fractional operator, addressing their efficacy in capturing non-local behaviors, accommodating power-law decay, and handling initial conditions. The discussion has also extended to the stability characteristics of solutions, offering insights into the suitability of these operators for diverse problem domains.

Through a synthesis of theoretical results and numerical simulations, we have highlighted the computational challenges associated with each fractional derivative. The findings presented here contribute to a comprehensive understanding of the strengths and limitations of fractional differential operators, guiding researchers and practitioners in the selection of appropriate tools for modeling and solving real-world problems.

This study sets the stage for further advancements in fractional calculus applications, with implications for improved modeling accuracy and enhanced computational efficiency. As we navigate the complexities of fractional calculus, we anticipate that this research will inspire future investigations and foster innovation in the evolving landscape of mathematical modeling and analysis.

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